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# MIRROR SYMMETRY ON K3 SURFACES AS A HYPERKÄHLER ROTATION

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Abstract. We show that under the hypotheses of [11], a mirror partner of a K3 surface $X$ with a fibration in special Lagrangian tori can be obtained by rotating the complex structure of $X$ within its hyperkähler family of complex structures. Furthermore, the same hypotheses force the B-field to vanish.

## 1. Introduction

According to the proposal of Strominger, Yau and Zaslow [11], the mirror partner of a K3 surface $X$ admitting a fibration in special Lagrangian tori should be identified with the moduli space of such fibrations (cf. also [7]). In more precise terms, the mirror partner $\check{X}$ should be identified with a suitable compactification of the relative Jacobian of $X'$, where $X'$ is an elliptic K3 surface obtained by rotating the complex structure of $X$ within its hyperkähler family of complex structures.

Morrison [10] suggested that such a compactification is provided by the moduli space of torsion sheaves of degree zero and pure dimension one supported by the fibers of $X'$. (It should be noted that whenever the fibration $X' \to \mathbb{P}^1$ admits a holomorphic section, as it is usually assumed in the physical literature, the complex manifolds $X'$ and $\check{X}$ turn out to be isomorphic). In [1] Morrison's suggestion was implemented, and it was shown that the relative Fourier-Mukai

transform defined by the Poincaré sheaf on the fiber product $X' \times_{\mathbb{P}^1} \check{X}$ enjoys some good properties related to mirror symmetry; e.g., it correctly maps D-branes in $X$ to D-branes in $\check{X}$, preserves the masses of the BPS states, etc. (The fact that the Fourier-Mukai transform might describe some aspects of mirror symmetry was already suggested in [3].)

It remains to check that $\check{X}$ is actually a mirror of $X$ in the sense of Dolgachev and Gross-Wilson, cf. [2, 5, 4]. In this note we show that this is indeed the case. Roughly speaking, we prove that whenever $X$ admits a fibration in special Lagrangian tori with a section, and also admits an elliptic mirror $\check{X}$ with a section,[1] then the complex structure of $\check{X}$ is obtained by that of $X$ by redefining the B-field and then performing a hyperkähler rotation. A more precise statement is as follows. Let $M$ be a primitive sublattice of the standard 2-cohomology lattice of a K3 surface, and denote by $\mathbf{K}_M$ the moduli space of pairs $(X, j)$, where $X$ is a K3 surface, and $j\colon M \to \mathrm{Pic}(X)$ is a primitive lattice embedding. Let $T = M^\perp$. We assume that $T$ contains a $U(1)$ lattice $P$; this means that the generic K3 surface $X$ in $\mathbf{K}_M$, possibly after a rotation of its complex structure within its hyperkähler family, admits a fibration in special Lagrangian tori with a section. After setting $\check{M} = T/P$, we assume that the generic K3 surface in $\mathbf{K}_{\check{M}}$ is elliptic and has a section. These hypotheses force the B-field to be an integral class. Then, by setting to zero this class (as it seems to be suggested by the physics, since in string theory the B-field is a class in $H^2(X, \mathbb{R}/\mathbb{Z})$), and rotating the complex structure of $X$ within its hyperkähler family of complex structures, we associate to $X \in \mathbf{K}_M$ a K3 surface $\check{X}$ in $\mathbf{K}_{\check{M}}$ such that $\mathrm{Pic}(\check{X}) \simeq \check{M}$.

## 2. Special Lagrangian fibrations and mirror K3 surfaces

We collect here, basically relying on [6, 9, 2, 5], some basic definitions and constructions about mirror families of K3 surfaces.

*Special Lagrangian submanifolds.* Let $X$ be an $n$-dimensional Kähler manifold with Kähler form $\omega$, and suppose that on $X$ there is a nowhere vanishing holomorphic $n$-form $\Omega$. One says that a real $n$-dimensional submanifold $\iota\colon Y \hookrightarrow X$ is *special Lagrangian* if $\iota^*\omega = 0$, and $\Omega$ can be chosen so that the form $\iota^*\Re e\,\Omega$ coincides with the volume form of $Y$. The moduli space of deformations of $Y$ through special Lagrangian submanifolds was described in [9].

Let $n = 2$, assume that $X$ is hyperkähler with Riemannian metric $g$, and choose basic complex structures $I$, $J$, and $K$. These generate an $S^2$ of complex structures compatible with the Riemannian metric of $X$, which we shall call the *hyperkähler family* of complex structures of $X$.

[1] These are the same assumptions made in [11] on physical grounds.

Denote by $\omega_I$, $\omega_J$ and $\omega_K$ the Kähler forms corresponding to the complex structures $I$, $J$ and $K$. The 2-form $\Omega_I = \omega_J + i\,\omega_K$ never vanishes, and is holomorphic with respect to $I$. Thus, submanifolds of $X$ that are special Lagrangian with respect to $I$, are holomorphic with respect to $J$ (this is a consequence of Wirtinger's theorem, cf. [6]). If $X$ is a complex K3 surface that admits a foliation by special Lagrangian 2-tori (in the complex structure $I$), then in the complex structure $J$ it is an elliptic surface, $p\colon X' \to \mathbb{P}^1$. If one wants $X$ to be compact then one must allow the fibration $p\colon X' \to \mathbb{P}^1$ to have some singular fibers, cf. [8].

*Mirror families of K3 surfaces* [2]. Let $L$ denote the lattice over $\mathbb{Z}$

$$L = U(1) \perp U(1) \perp U(1) \perp E_8 \perp E_8$$

(by "lattice over $\mathbb{Z}$" we mean as usual a free finitely generated $\mathbb{Z}$-module equipped with a symmetric $\mathbb{Z}$-valued quadratic form). If $X$ is a K3 surface, the group $H^2(X,\mathbb{Z})$ equipped with the cohomology intersection pairing is a lattice isomorphic to $L$.

If $M$ is an even nondegenerate lattice of signature $(1,t)$, a *$M$-polarized K3 surface* is a pair $(X,j)$, where $X$ is a K3 surface and $j\colon M \to \operatorname{Pic}(X)$ is a primitive lattice embedding. One can define a coarse moduli space $\mathbf{K}_M$ of $M$-polarized K3 surfaces; this is a quasi-projective algebraic variety of dimension $19-t$, and may be obtained by taking a quotient of the space

$$D_M = \left\{\mathbb{C}\Omega \in \mathbb{P}(M^\perp \otimes \mathbb{C}) \,|\, \Omega\cdot\Omega = 0, \Omega\cdot\bar\Omega > 0\right\}$$

by a discrete group $\Gamma_M$ (which is basically the group of isometries of $L$ that fix all elements of $M$) [2].

A basic notion to introduce the *mirror moduli space* to $\mathbf{K}_M$ is that of *admissible $m$-vector*. We shall consider here only the case $m=1$. Let us pick a primitive sublattice $M$ of $L$ of signature $(1,t)$.

**Definition 2.1.** *A 1-admissible vector $E \in M^\perp$ is an isotropic vector in $M^\perp$ such that there exists another isotropic vector $E' \in M^\perp$ with $E\cdot E' = 1$.*

After setting

$$\check M = E^\perp/\mathbb{Z}E$$

one easily shows that there is an orthogonal decomposition $M^\perp = P \oplus \check M$, where $P$ is the hyperbolic lattice generated by $E$ and $E'$. The orthogonal of $E$ is taken here in $M^\perp$. The *mirror moduli space* to $\mathbf{K}_M$ is the space $\mathbf{K}_{\check M}$. Of course one has

$$\dim \mathbf{K}_M + \dim \mathbf{K}_{\check M} = 20\,.$$

The operation of taking the "mirror moduli space" is a duality, i.e. $\check{\check M} \simeq M$ (this works so because we consider the case of a 1-admissible vector, and is no longer true for $m>1$).

*The interplay between special Lagrangian fibrations and mirror K3 surfaces.* Let again $M$ be an even nondegenerate lattice of signature $(1,t)$, and suppose that $X$ is K3 surface such that $\mathrm{Pic}(X) \simeq M$. The transcendental lattice $T$ (the orthogonal complement of $\mathrm{Pic}(X)$ in $H^2(X,\mathbb{Z})$) is an even lattice of signature $(2, 19-t)$. Let $\Omega = x + i\,y$ be a nowhere vanishing, global holomorphic two-form on $X$. Being orthogonal to all algebraic classes, the cohomology class of $\Omega$ spans a space-like 2-plane in $T \otimes \mathbb{R}$. The moduli space of K3 such that $\mathrm{Pic}(X) \simeq M$ is parametrized by the periods, whose real and imaginary parts are given by intersection with $x$ and $y$, respectively. Indeed, one should recall that if we fix a basis of the cohomology lattice $H^2(X,\mathbb{Z})$ given by integral cycles $\alpha_i$, $i = 1,\ldots,22$, every complex structure on $X$ is uniquely determined, via Torelli's theorem, by the complex valued matrix whose entries $\varpi_i$ are given by the intersections of the cycles $\alpha_i$ with the class of the holomorphic two-form $\Omega$, i.e. $\varpi_i = \alpha_i \cdot \Omega$. This shows that generically neither $x$ nor $y$ are integral classes in the cohomology ring. However, if we make the further request that there is a 1-admissible vector in $T$, and make some choices, one of the two classes is forced to be integral.

We recall now a result from [5] (although in a slightly weaker form).

**Proposition 2.2.** *There exists in $T$ a 1-admissible vector if and only if there is a complex structure on $X$ such that $X$ has a special Lagrangian fibration with a section.*

So we consider on $X$ a complex structure satisfying this property (it follows from [5] that, if we fix a hyperkähler metric on $X$, this complex structure belongs to the same hyperkähler family as the one we started from). As a direct consequence we have

**Proposition 2.3.** *If there exists a 1-admissible vector in $T$ one can perform a hyperkähler rotation of the complex structure and choose a nowhere vanishing two-form $\Omega$, holomorphic in the new complex structure, whose real part $\Re e\,\Omega$ is integral.*

*Proof.* By Proposition 1.3 of [5] the existence of a 1-admissible vector implies the existence on $X$ of a special Lagrangian fibration with a section. On the other hand by [6] what is special Lagrangian in a complex structure is holomorphic in the complex structure in which the Kähler form is given by $\Re e\,\Omega$. Thus in this complex structure the Picard group is nontrivial, which implies that the surface is algebraic, i.e. $\Re e\,\Omega$ is integral. □

## 3. The construction

We introduce now a moduli space $\tilde{\mathbf{K}}_M$ parametrizing $M$-polarized K3 surfaces together with of a 1-admissible vector in $T = M^\perp$. The generic K3

surface $X$ in $\tilde{\mathbf{K}}_M$ admits a fibration in special Lagrangian tori with a section; the primitive $U(1)$ sublattice $P$ of the transcendental lattice $T$ associated with the 1-admissible vector is generated by the class of the fiber and the class of the section. We fix a marking[2] of $X$, i.e., a lattice isomorphism $\psi\colon H^2(X,\mathbb{Z}) \to L$. We have an isomorphism

$$L \simeq M \oplus P \oplus \check{M},$$

where $\check{M} = T/P$. The fact that $\check{\check{M}} \simeq M$ implies that the moduli spaces $\tilde{\mathbf{K}}_M$ and $\tilde{\mathbf{K}}_{\check{M}}$ are isomorphic. Generically, we may assume that $M \simeq \psi(\mathrm{Pic}(X))$.

One easily shows that the following assumptions are generically equivalent to each other (where "generically" means that this holds true for $X$ in a dense open subset of $\tilde{\mathbf{K}}_M$):

(i) The lattice $\check{M}$ contains a primitive $U(1)$ sublattice $P'$.

(ii) The generic K3 surface in the mirror moduli space $\mathbf{K}_{\check{M}}$ is an elliptic fibration with a section.

(iii) $X$ carries two fibrations in special Lagrangian tori admitting a section, in such a way that the corresponding $U(1)$ lattices $P$, $P'$ are orthogonal.[3]

The two $U(1)$ lattices $P$ and $P'$ are interchanged by an isometry of $L$. Thus, the operation of exchanging them has no effect on the moduli space $\mathbf{K}_M$ (although it does on $D_M$).

We shall assume one of these equivalent conditions. The form (ii) of the second condition shows that we are working exactly under the same assumptions that in [11] are advocated on physical grounds.

In the complex structure of $X$ we have fixed at the outset we have the Kähler form $\omega$ and the holomorphic two-form $\Omega = x + i\,y$, with $x$ an integral class. Condition (iii) means that $P'$ is calibrated by $x$. If we perform a rotation around the $y$ axis, mapping the pair $(\omega, x)$ to $(x, -\omega)$, we still obtain an algebraic K3 surface $X'$ whose Picard group contains $P'$ [5].

Now we want to show that the Kähler class of $X'$ is a space-like vector contained in the hyperbolic lattice $P'$. We remind here that the explicit mirror map in [2] and [5] is given in terms of a choice of a hyperbolic sublattice of the transcendental lattice. Let $D_M$ be defined as in Section 2, and let

$$T_M = \{B + i\,\omega \in M \otimes \mathbb{C} \,|\, \omega \cdot \omega > 0\} = M \times V(M)^+\,.$$

Here $V(M)^+$ is the component of the positive cone in $M \otimes \mathbb{R}$ that contains the Kähler form of $X$. The space $T_M$ can be regarded as a (covering of the) moduli space of " complexified Kähler structures" on $X$. Let $M' = T/P' \simeq \check{M}$. By [5] Proposition 1.1, the mirror map is an isomorphism

$$\phi\colon T_{M'} \to D_M\,,$$

[2]Since we are fixing a marking of $X$ in the following we shall often confuse the lattices $H^2(X,\mathbb{Z})$ and $L$.

[3]Then one shows that the direct sum $P \oplus P'$ is an orthogonal summand of $T$.

$$\phi(\check{B}+i\check{\omega}) = \check{B} + E' + \tfrac{1}{2}(\check{\omega}\cdot\check{\omega} - \check{B}\cdot\check{B})E + i\,(\check{\omega} - (\check{\omega}\cdot\check{B})E)\,.$$

Here $E$ and $E'$ are the two isotropic generators of the $U(1)$ lattice $P'$, while $\check{B}$ is what the physicists call the B-field. Our holomorphic two-form $\Omega$ is of course of the form $\phi(\check{B}+i\check{\omega})$ for suitable $\check{B}$ and $\check{\omega}$, since $\phi$ is an isomorphism. The Kähler class of $X'$ is given by

$$x = \Re e\,\Omega = \check{B} + E' + \tfrac{1}{2}(\check{\omega}\cdot\check{\omega} - \check{B}\cdot\check{B})E$$

and the new global holomorphic two-form is $-\omega + iy$. Since $\check{B}$ is orthogonal to $E$ and $E'$, it is an integral class.

However, the Picard lattice of the K3 surface $X'$ is generically not isomorphic to $\check{M}$. A better choice is suggested by the physics. Indeed in most string theory models the B-field is regarded as a Chern-Simons term, namely, as a class in $H^2(X,\mathbb{R}/\mathbb{Z})$; so, if we consider the projection $\lambda\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R}/\mathbb{Z})$, the relevant moduli space should be

$$\tilde{T}_{M'} = \lambda(M'\otimes\mathbb{R}) \times V(M')^+$$

instead of $T_{M'}$. To take this suggestion into account we set $\check{B}=0$. Since $y = \check{\omega} - (\check{\omega}\cdot\check{B})E$, this changes the complex structure in $X'$. Moreover, $x$ lies now in $P'$.

So, let us now consider the intersection of $P\otimes\mathbb{R}$ with the spacelike two-plane $\langle\Omega\rangle$ spanned by $\Omega$. This cannot be trivial, since $P$ is hyperbolic and $T\otimes\mathbb{R}$ is of signature $(2, 19-t)$. So we have a real space-like class in $P\otimes\mathbb{R}\cap\langle\Omega\rangle$ that is orthogonal to $x$ by construction and thus must be equal (up to a scalar factor) to $y$. But then, in the complex structure in which the Kähler form is given by $x$, all the cycles of $\check{M}$ are orthogonal to the new holomorphic two-form, given by $\omega + iy$, and therefore are algebraic. (Notice that the class $y$ is not integral.)

## 4. Conclusions

A first conclusion we may draw is that the hypotheses of [11] force the B-field to be integral, namely, to be zero as a class $\check{B}\in H^2(X,\mathbb{R}/\mathbb{Z})$. Moreover, starting from a K3 surface $X$ in $\tilde{\mathbf{K}}_M$, the construction in the previous section singles out a point in the variety $\tilde{\mathbf{K}}_{\check{M}}$; so we have established a map

$$\mu\colon \tilde{\mathbf{K}}_M \to \tilde{\mathbf{K}}_{\check{M}}$$

which is bijective by construction, and deserves to be the called *the mirror map*. This map consists in setting $\check{B}$ to zero (as a class in $H^2(X,\mathbb{Z})$) and then performing a hyperkähler rotation.

If we do not set $\check{B}$ to zero, we obtain a family of K3 surfaces, labelled by the possible values of $\check{B}\in M'\simeq\check{M}$. Its counterpart under mirror symmetry is a family of K3 surfaces labelled by $M$. The two families are related by a hyperkähler rotation.

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